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On homogeneity of discrete-time systems: stability and convergence rates

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Abstract

A new definition of homogeneity for discrete-time systems is introduced. As in the continuous-time case, the property can be verified algebraically in the transition map of the system, and it implies that a dilation of the initial conditions leads to a scaling of the trajectory. Stability properties and convergence rates of the system's solutions can be established by considering only the homogeneity degree. The existence of homogeneous Lyapunov and Lyapunov-like functions is proven.

KEYWORDS:
Discrete-time systems, Lyapunov methods, Homogeneous systems, Nonlinear systems.

1 | INTRODUCTION

Owing to the advantages that the homogeneity offers in the analysis and design of continuous-time (CT) control systems, it has remarkably been studied by many authors [1,2,3,4,5,6,7,8,9,10,11]. Some of the main features of homogeneous systems are:

1. the homogeneity is easily verifiable in the vector field of the system;
2. the trajectories of a homogeneous system exhibit a dilating property, i.e. a dilation of the initial condition is equivalent to a scaling of the original trajectory, and as a consequence, local properties of homogeneous systems can be translated into global ones;
3. when the origin of a homogeneous system is asymptotically stable, the convergence rate is specified by the homogeneity degree;
4. there exist converse Lyapunov theorems that assert the existence of smooth homogeneous Lyapunov functions (LF) for homogeneous system;
5. the homogeneity degree allows to determine several robustness properties of the system with respect to external inputs, delays and unmodeled dynamics [12,13].

Discrete-time (DT) systems, in their turn, are of great importance in Control Systems Theory. They arise as mathematical models of real processes, e.g. in Economics, Biology, and Engineering [14]. DT systems are also obtained by means of discretization of CT systems. Hence, many of the concepts and tools for analysis and design of CT control systems have been transferred to the DT counterpart, see e.g. [15,16,14,17,18]. Naturally, the advantages of homogeneity for CT systems are desired to be extended to the DT case, however, as mentioned in [17], the extension of the CT properties to the DT case require modifications that are not completely obvious.

Some approaches to extend homogeneity to DT systems have been proposed in [19] and [20]. The definition of homogeneity for DT systems given in [19] is basically the same as for CT systems. Consequently, for the DT systems, only the case of homogeneity of degree zero has the benefits of homogeneity present in the CT case. In [20] a wider family of dilations is defined, however, all the developments are restricted to the case of homogeneity of degree zero. Those results indicate that homogeneity of degree zero, is the only case when the definition for CT systems can be applied directly in the DT case by preserving the advantages from the CT framework. Hence, some more suitable concepts of homogeneity for DT systems should be introduced.

In this paper we introduce a new definition of homogeneity for DT systems, we refer to it as $D_{\mathbf{r}}-$homogeneity. The following properties are proven for $D_{\mathbf{r}}-$homogeneous DT systems:

1. $D_{\mathbf{r}}-$homogeneity can be verified algebraically in the transition map of the DT system;
2. the trajectories of a $D_{\mathbf{r}}-$homogeneous system admit dilation, i.e. a scaling of the initial condition is equivalent to a scaling of the whole trajectory;
3. we show that $D_{\mathbf{r}}-$homogeneity directly implies local asymptotic stability of the system's origin or global boundedness of the trajectories depending only on the $D_{\mathbf{r}}-$homogeneity degree;
4. when the origin is locally asymptotically stable, the trajectories starting far enough from the origin are unbounded. In the case of global boundedness the origin is locally unstable;
5. convergence rates of system's solutions can be established by considering only the homogeneity degree. In particular, when the origin is asymptotically stable, the convergence rate of the trajectories can be exponential or hyper-exponential;
6. the existence of homogeneous Lyapunov and Lyapunov-like functions is guaranteed in every case. Some of the results of this paper were briefly announced in [21] without proofs.

Paper organization: In Section 2, the definition of $D_{\mathbf{r}}-$homogeneity is stated. The stability properties of $D_{\mathbf{r}}-$homogeneous systems are described in Section 3. Convergence rates are analysed in Section 4. Examples of the theoretical results are given in Section 5. Some concluding remarks and future research are stated in Section 6.

Notation: Real and integer numbers are denoted as $\mathbb{R}$ and $\mathbb{Z}$, respectively. $\mathbb{R}_{>0}$ denotes the set $\{x \in \mathbb{R} \ : \ x > 0\}$, analogously for the set $\mathbb{Z}$ and the sign $\geq$. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and $\|x\|_{\mathbf{r}}$ a $\mathbf{r}$-homogeneous norm (see Appendix B). The composition of two functions f and g (with adequate domains and codomains) is denoted as $f \circ g$, i.e. $(f \circ g)(x) = f(g(x))$. Thus, we denote $(\underbrace{f \circ f \circ \cdots \circ f}_{r-\text{times}})(x) = f^{[r]}(x)$, for example, $f^{[1]}(x) = f(x)$ and $f^{[2]}(x) = f(f(x))$. For a continuous positive definite function $V \ : \ \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and some $\alpha, \bar{\alpha} \in \mathbb{R}_{>0}$, $\mathcal{I}(V, \alpha)$ and $\mathcal{E}(V, \bar{\alpha})$ denote the sets $\{x \in \mathbb{R}^n \ : \ V(x) \leq \alpha\}$ and $\{x \in \mathbb{R}^n \ : \ V(x) \geq \bar{\alpha}\}$, respectively. For $x \in \mathbb{R}$ and $q \in \mathbb{R}_{>0}$, $\lceil x \rfloor^q$ denotes $\operatorname{sign}(x)|x|^q$.

2 | $D_{\mathbf{r}}-$HOMOGENEITY

We consider an autonomous DT dynamic system

$$x(k+1) = f(x(k)), \tag{1}$$

where the state $x(k) \in \mathbb{R}^n$, for any $k \in \mathbb{Z}_{\geq 0}$. We assume that the transition map $f \ : \ \mathbb{R}^n \to \mathbb{R}^n$ is well defined and locally bounded for all $x \in \mathbb{R}^n$. Such an assumption is sufficient to guarantee existence and uniqueness of solutions, see e.g. [14] p. 5. Let $F(k; x_0)$ denote the solution of (1) with initial condition $x_0 = x(0)$, thus, $F(0; x_0) = x_0$. Note that

$$F(1; x_0) = f(x_0), \quad F(2; x_0) = f(F(1; x_0)), \quad \ldots$$

Therefore, $F(k; x_0) = f^{[k]}(x_0)$. In this notation, $F(0; x_0) = f^{[0]}(x_0) = x_0$.

Before stating the definition of $D_{\mathbf{r}}-$homogeneity, let us recall the usual definition of weighted homogeneity.

Definition 1 ([22]). Let $\Lambda_\epsilon^{\mathbf{r}}$ denote the family of dilations given by the square diagonal matrix $\Lambda_\epsilon^{\mathbf{r}} = \operatorname{diag}(\epsilon^{r_1}, \ldots, \epsilon^{r_n})$, where $\mathbf{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$. The components of $\mathbf{r}$ are called the *weights* of the coordinates. Thus:

a) a function $V \ : \ \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}$ if $V\left(\Lambda_\epsilon^{\mathbf{r}} x\right) = \epsilon^m V(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$;

b) a vector field $f \ : \ \mathbb{R}^n \to \mathbb{R}^n$, $f = [f_1, \ldots, f_n]^\top$, is $\mathbf{r}$-homogeneous of degree $\kappa \in \mathbb{R}$ if for each $i = 1, \ldots, n$, $f_i\left(\Lambda_\epsilon^{\mathbf{r}} x\right) = \epsilon^{\kappa + r_i} f_i(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$.

Remark 1. Definition 1 is used in[19] for DT systems for the case $\kappa = 0$. The definition given in[20] coincides with Definition 1 for the case of dilations $\Lambda_\epsilon^{\mathbf{r}}$, and also considers only the case $\kappa = 0$. The main reason to restrict the homogeneity degree to $\kappa = 0$ is the following. The solution of a DT system is computed by the successive composition of its transition map. Thus, if in (1) f is $\mathbf{r}$-homogeneous of degree κ, then

$$f\left(f(\Lambda_\epsilon^{\mathbf{r}} x_0)\right) = f\left(\epsilon^\kappa \Lambda_\epsilon^{\mathbf{r}} f(x_0)\right) = \epsilon^\kappa \Lambda_\epsilon^{\mathbf{r}} f(\epsilon^\kappa f(x_0)) .$$

Hence, it is clear that the scaling property for the system's solution can be obtained only for the case $\kappa = 0$.

$D_{\mathbf{r}}-$homogeneity consists in a modification of the usual definition for vector fields, we state it as follows.

Definition 2 ($D_{\mathbf{r}}-$homogeneity)**.** Let $\Lambda_\epsilon^{\mathbf{r}}$, $\mathbf{r}$ and ϵ be as in Definition 1. A map $f : \mathbb{R}^n \to \mathbb{R}^n$, $f = [f_1, \ldots, f_n]^\top$, is $D_{\mathbf{r}}$-homogeneous of degree ν if for each $i = 1, \ldots, n$, $f_i\left(\Lambda_\epsilon^{\mathbf{r}} x\right) = \epsilon^{r_i \nu} f_i(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$, and some $\nu \in \mathbb{R}_{>0}$, or equivalently, $f(\Lambda_\epsilon^{\mathbf{r}} x) = (\Lambda_\epsilon^{\mathbf{r}})^\nu f(x) = \Lambda_\epsilon^{\nu\mathbf{r}} f(x) = \Lambda_{\epsilon^\nu}^{\mathbf{r}} f(x)$.

We say that (1) is $D_{\mathbf{r}}-$homogeneous of degree ν if its transition map f is $D_{\mathbf{r}}-$homogeneous of degree ν. Observe that we are not modifying the definition for scalar functions. Also note that, for vector fields and maps, both definitions are equivalent if and only if $\kappa = 0$ and $\nu = 1$, respectively. Thus, the case of $D_{\mathbf{r}}-$homogeneity of degree one for (1) coincides with the case of $\mathbf{r}-$homogeneity of degree zero given in[19] and[20]. The following lemma states some properties of $\mathbf{r}-$homogeneous functions and $D_{\mathbf{r}}-$homogeneous vector fields.

Lemma 1. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous $\mathbf{r}-$homogeneous function of degree m. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $D_{\mathbf{r}}-$homogeneous map of degree ν. Then:

a) $V \circ f$ is an $\mathbf{r}-$homogeneous function of degree $\bar{m} = \nu m$;

b) $V \circ f$ is locally bounded if f is locally bounded, and $V \circ f$ is continuous if f is continuous;

c) $V \circ f$ is positive semidefinite if V is positive semidefinite;

d) $V \circ f$ is positive definite if V is positive definite and f is such that $f(x) = 0$ if and only if $x = 0$;

e) for any $k \in \mathbb{Z}_{>0}$, $f^{[k]}$ is $D_{\mathbf{r}}$-homogeneous of degree $\bar{\nu} = \nu^k$.

Proof. a) By using Definition 2 and Definition 1, we have that $V(f(\Lambda_\epsilon^{\mathbf{r}} x)) = V((\Lambda_\epsilon^{\mathbf{r}})^\nu f(x)) = V(\Lambda_{\epsilon^\nu}^{\mathbf{r}} f(x)) = \epsilon^{\nu m} V(f(x))$. Therefore $V(f(x))$ is $\mathbf{r}-$homogeneous of degree $\bar{m} = m\nu$. Points b), c), and d) are straightforward. e) According to Definition 2, $f^{[k]}(\Lambda_\epsilon^{\mathbf{r}} x) = f^{[k-1]}\left((\Lambda_\epsilon^{\mathbf{r}})^\nu f(x)\right) = f^{[k-2]}\left((\Lambda_\epsilon^{\mathbf{r}})^{\nu^2} f^{[2]}(x)\right) = \cdots = (\Lambda_\epsilon^{\mathbf{r}})^{\nu^k} f^{[k]}(x)$. □

Let us recall the result about the scaling property of the solutions of CT $\mathbf{r}-$homogeneous systems.

Lemma 2 (see e.g.[1],[2])**.** Consider the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, and suppose that f is $\mathbf{r}-$homogeneous of degree κ. Let $\phi(t; x_0)$ denote the solution of the system with initial condition $x_0 = x(0)$, then

$$\phi(t; \Lambda_\epsilon^{\mathbf{r}} x_0) = \Lambda_\epsilon^{\mathbf{r}} \phi(\epsilon^\kappa t; x_0) .$$

The extension of this result to homogeneous differential inclusions is in[8],[11]. Now we give an analogous result for $D_{\mathbf{r}}-$homogeneous DT systems [1].

Lemma 3. If (1) is $D_{\mathbf{r}}-$homogeneous of degree ν, then its solutions $F(k; x_0)$ are such that

$$F(k; \Lambda_\epsilon^{\mathbf{r}} x_0) = (\Lambda_\epsilon^{\mathbf{r}})^{\nu^k} F(k; x_0) , \quad \forall k \in \mathbb{Z}_{\geq 0} . \tag{2}$$

Proof. For $k = 0$, $F(0; \Lambda_\epsilon^{\mathbf{r}} x_0) = \Lambda_\epsilon^{\mathbf{r}} x_0$. For $k \in \mathbb{Z}_{>0}$, $F(k; \Lambda_\epsilon^{\mathbf{r}} x_0) = f^{[k]}(\Lambda_\epsilon^{\mathbf{r}} x_0)$, hence, by using Lemma 1 we obtain $f^{[k]}(\Lambda_\epsilon^{\mathbf{r}} x_0) = (\Lambda_\epsilon^{\mathbf{r}})^{\nu^k} f^{[k]}(x_0) = (\Lambda_\epsilon^{\mathbf{r}})^{\nu^k} F(k; x_0)$. □

Remark 2. As $\mathbf{r}-$homogeneous CT systems, $D_{\mathbf{r}}-$homogeneous DT systems admit scaling of their trajectories.

[1] In[21], there is an error in the corresponding equation to (2), it says $(\Lambda_\epsilon^{\mathbf{r}})^{k\nu}$ instead of $(\Lambda_\epsilon^{\mathbf{r}})^{\nu^k}$.

3 | STABILITY PROPERTIES OF $D_{\mathbf{r}}$−HOMOGENEOUS SYSTEMS

In this section, (1) is assumed to be $\mathbf{D}_{\mathbf{r}}$−homogeneous of some degree ν with a unique equilibrium point at the origin. We show that the value of ν directly implies stability features of (1). All the results in this section are Lyapunov-based. Thus, although the definitions of stability in the sense of Lyapunov for DT systems coincide, in general, with those for CT systems, we have included them in Appendix A. We evoke below the characterization of stability by means of LFs, for this purpose, we recall that $x \in \mathbb{R}^n$ is an equilibrium point of (1) if it is a solution of the equation $f(x) - x = 0$. For the following theorem see e.g.[2],[14] (and the references therein), see also[23] Theorem 14 and Remark 11.

Theorem 1. Let $x = 0$ be an equilibrium point of (1) and let $\mathcal{S} \subset \mathbb{R}^n$ be a neighbourhood of $x = 0$. If there exist a continuous positive definite function $V : \mathcal{S} \to \mathbb{R}$ such that

- $\Delta V(x) := V(f(x)) - V(x) \leq 0$ for all $x \in \mathcal{S}$, then $x = 0$ is a stable equilibrium point of (1),
- $\Delta V(x) \leq -\eta(|x|)$, for all $x \in \mathcal{S}$, and some continuous positive definite function η, then $x = 0$ is an asymptotically stable equilibrium point of (1).

The following simple but useful result is required to guarantee the existence of LFs for $\mathbf{D}_{\mathbf{r}}$−homogeneous systems.

Lemma 4. For any $n \in \mathbb{Z}_{>0}$, any $m \in \mathbb{R}_{>0}$, and any $\mathbf{r} \in \mathbb{R}_n$, $r_i > 0$, $i = 1, \dots, n$, the function $V : \mathbb{R}^n \to \mathbb{R}$ given by $V(x) = \sum_{i=1}^{n} \beta_i |x_i|^{\frac{m}{r_i}}$, $\beta_i \in \mathbb{R}_{>0}$ is continuous, positive definite, and $\mathbf{r}$−homogeneous of degree m. Moreover, if $m > \max_i\{r_i\}$, then V is differentiable for all $x \in \mathbb{R}^n$.

Proof. The smoothness properties of V are deduced directly from the smoothness properties of the function $|x_i|^{\frac{m}{r_i}}$. □

For some of the results we require the following assumption.

A1 The transition map f of (1) is such that $\inf_{y \in S} |f(y)| > 0$, where $S = \{x \in \mathbb{R}^n \ : \ |x| = 1\}$.

Note that: 1) to fulfil **A1**, it is necessary (by homogeneity) that $f(x) = 0 \Rightarrow x = 0$; 2) **A1** is always satisfied by a continuous map such that $f(x) = 0 \Rightarrow x = 0$.

3.1 | $D_{\mathbf{r}}$−homogeneous systems of degree $\nu > 1$

For the following theorem, we use the sets $\mathcal{I}$ and $\mathcal{E}$ defined in the notation given in Section 1.

Theorem 2. Suppose that (1) is $\mathbf{D}_{\mathbf{r}}$-homogeneous of degree $\nu > 1$, with a unique equilibrium point at $x = 0$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive definite $\mathbf{r}$-homogeneous function of degree $m \in \mathbb{R}_{>0}$. Then:

1) $x = 0$ is a locally asymptotically stable equilibrium point of (1), and there exists $\alpha \in \mathbb{R}_{>0}$ such that V is a LF for (1) on $\mathcal{I}(V, \alpha)$;
2) if in addition f satisfies **A1**, then there exists $\bar{\alpha} \in \mathbb{R}_{>0}$ such that for all $x_0 \in \mathcal{E}(V, \bar{\alpha})$, $|F(k; x_0)| \to \infty$ as $k \to \infty$.

Proof. 1) From Lemma 1, the function $V \circ f$ is nonnegative, and $\mathbf{r}$-homogeneous of degree $\bar{m} = m\nu$. Therefore (see Lemma 7 in Appendix B), there exists $\overline{\gamma} \in \mathbb{R}_{>0}$ such that

$$V(f(x)) \leq \overline{\gamma} V^{\nu}(x), \quad \forall x \in \mathbb{R}^n . \tag{3}$$

Define $\alpha = \left(\frac{1-\varepsilon}{\overline{\gamma}}\right)^{\frac{1}{\nu-1}}$, for any $\varepsilon \in (0, 1)$. Thus,

$$\Delta V(x) \leq \overline{\gamma} V^{\nu}(x) - V(x) = \left(\overline{\gamma} V^{\nu-1}(x) - 1\right) V(x) .$$

Since $V(x) \leq \alpha$ for all $x \in \mathcal{I}(V, \alpha)$,

$$\Delta V(x) \leq \left(\overline{\gamma}\alpha^{\nu-1} - 1\right) V(x) = -\varepsilon V(x) .$$

From Theorem 1 we conclude the proof for the first part of Theorem 2.

2) For the second one, Lemma 6 assures that there exists $\underline{\gamma} \in \mathbb{R}_{>0}$ such that

$$V(f(x)) \geq \underline{\gamma} V^{\nu}(x), \quad \forall x \in \mathbb{R}^n . \tag{4}$$

Define $\bar{\alpha} = \left(\frac{\varepsilon+1}{\underline{\gamma}}\right)^{\frac{1}{\nu-1}}$, for any $\varepsilon \in \mathbb{R}_{>0}$. Thus, $\Delta V(x) \geq \underline{\gamma} V^{\nu}(x) - V(x) = \left(\underline{\gamma} V^{\nu-1}(x) - 1\right) V(x)$. Since $V(x) \geq \bar{\alpha}$ for all $x \in \mathcal{E}(V, \bar{\alpha})$,

$$\Delta V(x) \geq \left(\underline{\gamma} \bar{\alpha}^{\nu-1} - 1\right) V(x) = \varepsilon V(x) .$$

Thus, for each $k \geq 0$, $V(F(k+1; x_0)) > V(F(k; x_0))$. Now

$$\sum_{j=0}^{k} \Delta V(x(j)) = V(F(k; x_0)) - V(x_0) \geq \sum_{j=0}^{k} \varepsilon V(x(j)) \geq k\varepsilon\bar{\alpha} ,$$

hence, $V(F(k; x_0)) \geq V(x_0) + k\varepsilon\bar{\alpha}$. Since V is well-defined for all $x \in \mathbb{R}^n$, continuous, $\mathbf{r}$-homogeneous and positive definite, then it is radially unbounded [10] Lemma 4.1, thus,

$$k \to \infty \;\Rightarrow\; V(F(k; x_0)) \to \infty \;\Rightarrow\; |x(k)| \to \infty . \qquad \square$$

From Theorem 2 we can make the following observations:

- To ensure (local) asymptotic stability of the origin of (1), it is sufficient to verify $D_{\mathbf{r}}$−homogeneity of degree $\nu > 1$.
- Theorem 2 guarantees that any $\mathbf{r}$-homogeneous positive definite function V of degree $m \in \mathbb{R}_{>0}$ is a LF for (1).
- The proof of the theorem gives a procedure to estimate the attraction domain as $\mathcal{I}(V, \alpha)$.
- $D_{\mathbf{r}}$−homogeneity degree $\nu > 1$ only guarantees local asymptotic stability of $x = 0$. Nonetheless, under assumption **A1**, the trajectories starting far enough from the origin cannot remain bounded (despite the origin is asymptotically stable).

3.2 | $D_{\mathbf{r}}$−homogeneous systems of degree $\nu \in (0, 1)$

For the definition of practical stability used in the following theorem see [14] or Appendix A.

Theorem 3. Suppose that (1) is $D_{\mathbf{r}}$-homogeneous of degree $\nu \in (0, 1)$ with a unique equilibrium point at $x = 0$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive definite $\mathbf{r}$-homogeneous function of degree $m \in \mathbb{R}_{>0}$. Then:

1) the origin of (1) is globally practically stable, and there exists $\bar{\alpha} \in \mathbb{R}_{>0}$ such that $\Delta V(x) < 0$ for all $x \in \mathcal{E}(V, \bar{\alpha})$;

2) if f satisfies **A1**, the origin of (1) is locally unstable.

Proof. 1) From the assumptions on f and V, Lemma 7 guarantees that there exists $\overline{\gamma} \in \mathbb{R}_{>0}$ such that (3) holds. Define $\bar{\alpha} = (\varepsilon + \overline{\gamma})^{\frac{1}{1-\nu}}$ for any $\varepsilon \in \mathbb{R}_{>0}$. Thus, for all $x \in \mathcal{E}(V, \bar{\alpha})$,

$$\Delta V(x) \leq \overline{\gamma} V^{\nu}(x) - V(x) = \left(\overline{\gamma} - V^{1-\nu}(x)\right) V^{\nu}(x) \leq \left(\overline{\gamma} - \bar{\alpha}^{1-\nu}\right) V^{\nu}(x) = -\varepsilon V^{\nu}(x) .$$

Note that $-\varepsilon V^{\nu}(x) \leq -\varepsilon\bar{\alpha}^{\nu}$ for all $x \in \mathcal{E}(V, \bar{\alpha})$. Thus, according to [14] Theorem 5.14.2, Corollary 5.14.3, V proves global practical stability of the origin of (1).

2) According to Lemma 7 there exists $\underline{\gamma} \in \mathbb{R}_{>0}$ such that (4) holds. Define $\alpha = (\underline{\gamma} - \varepsilon)^{\frac{1}{1-\nu}}$ for any $\varepsilon \in \mathbb{R}_{>0}$, $\varepsilon < \underline{\gamma}$. Thus, for all $x \in \mathcal{I}(V, \alpha)$,

$$\Delta V(x) \geq \underline{\gamma} V^{\nu}(x) - V(x) = \left(\underline{\gamma} - V^{1-\nu}(x)\right) V^{\nu}(x) \geq \left(\underline{\gamma} - \alpha^{1-\nu}\right) V^{\nu}(x) = \varepsilon V^{\nu}(x) .$$

Thus, from [14] Theorem 5.9.3 we have that $x = 0$ is an unstable equilibrium point of (1). $\square$

From Theorem 3 we can observe the following:

- $D_{\mathbf{r}}$−homogeneity of degree $\nu \in (0, 1)$ guarantees that, for any bounded initial condition, the system's trajectories reach a vicinity of the origin in a finite number of steps.
- Such *stability* property can be verified through any continuous positive definite $\mathbf{r}$-homogeneous function V of any degree $m \in \mathbb{R}_{>0}$.
- The proof of the theorem gives a procedure to estimate the practical stability-region $\mathcal{S}$, note that $\mathcal{S} \subset \mathbb{R}^n \setminus \mathcal{E}(V, \bar{\alpha})$.

- $D_{\mathbf{r}}-$homogeneity of degree $\nu \in (0,1)$ prevents global asymptotic stability of the origin when f satisfies **A1**, note that in this case, there are no solutions converging to the origin.

To finalize this section we state a robustness property of $D_{\mathbf{r}}-$homogeneous systems. Consider the disturbed DT dynamic system

$$x(k+1) = f(x(k)) + \delta(k)\,, \tag{5}$$

where x and f are as in (1). The disturbance $\delta : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ is such that $|\delta(k)| \leq \bar{\delta}$, for all $k \in \mathbb{Z}_{\geq 0}$, for some $\bar{\delta} \in \mathbb{R}_{\geq 0}$.

Theorem 4. Suppose that, in (5), f is $D_{\mathbf{r}}$-homogeneous of degree $\nu \in (0,1)$, and $x = 0$ is the unique solution of the equation $f(x) - x = 0$. For any finite $\bar{\delta} \in \mathbb{R}_{\geq 0}$, the origin of (5) is globally practically stable.

Proof. Consider any positive definite $\mathbf{r}$-homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of some degree $m \in \mathbb{R}_{>0}$ such that V is $\alpha-$Hölder continuous[2] for any compact set of $\mathbb{R}^n$ (such a function always exists, e.g. in the class of functions defined in Lemma 4 if $m \leq \min_i(r_i)$). Note that $\Delta V(x) = V(f(x)+\delta) - V(x)$ can be rewritten as $\Delta V(x) = V(f(x)) - V(x) + V_\delta(x)$ where $V_\delta(x) = V(f(x)+\delta) - V(f(x))$. Once more, we use Lemma 7 to assure that there exists $\overline{\gamma} \in \mathbb{R}_{>0}$ such that (3) holds. Define $\bar{\alpha} = (\varepsilon + \overline{\gamma})^{\frac{1}{1-\nu}}$ for any $\varepsilon \in \mathbb{R}_{>0}$. By following an analogous procedure as in the proof of Theorem 3 we have that, for any $x \in \mathcal{E}(V, \bar{\alpha})$, $\Delta V(x) \leq -\varepsilon V^\nu(x) + |V_\delta(x)|$. Now, according to Lemma 8 there exists $L_{\bar{\delta}} \in \mathbb{R}_{>0}$ such that

$$|V_\delta(x)| \leq \begin{cases} L_{\bar{\delta}}|\delta|^\alpha & \text{if } \|f(x)\|_{\mathbf{r}} \leq 1\,, \\ L_{\bar{\delta}}\|f(x)\|_{\mathbf{r}}^\rho |\delta|^\alpha & \text{if } \|f(x)\|_{\mathbf{r}} \geq 1\,, \end{cases}$$

where $\rho = m - \alpha \min_i\{r_i\}$. From Lemma 1, the function $x \mapsto \|f(x)\|_{\mathbf{r}}^\rho$ is nonnegative and $\mathbf{r}-$homogeneous of degree $\nu\rho$. Thus, by using Lemmas 5 and 6, we obtain $L_{\bar{\delta}}\|f(x)\|_{\mathbf{r}}^\rho \leq \eta L_{\bar{\delta}} V^{\frac{\nu\rho}{m}}(x)$ for some $\eta \in \mathbb{R}_{>0}$. Therefore, $\Delta V(x) \leq -(1-\theta)\varepsilon V^\nu(x)$, for any $\theta \in (0,1)$, for all $x \in \mathcal{E}(V, M)$, where $M = \max\left\{ \bar{\alpha}, \left(\frac{L_{\bar{\delta}}\bar{\delta}^\alpha}{\theta\varepsilon}\right)^{\frac{1}{\nu}}, \left(\frac{\eta L_{\bar{\delta}}\bar{\delta}^\alpha}{\theta\varepsilon}\right)^{\frac{m}{\nu(m-\rho)}} \right\}$. Thus, according to[14] Theorem 5.14.2, Corollary 5.14.3, V proves practical stability of the origin of (5). □

3.3 | $D_{\mathbf{r}}-$homogeneous systems of degree $\nu = 1$

We have seen that for $D_{\mathbf{r}}-$homogeneity degrees $\nu > 1$ and $\nu \in (0,1)$ there cannot be global asymptotic stability of the system's origin if f satisfies **A1**. However, the proven stability properties only depend on the $D_{\mathbf{r}}-$homogeneity degree of the system, and not on any other features as e.g. on system's parameters. On the other hand, for the case $\nu = 1$, global asymptotic stability is possible, for example, if (1) is linear, then it is $D_{\mathbf{r}}-$homogeneous of degree $\nu = 1$ and the asymptotic stability property relies on the values of the system's parameters. It is important to mention that when (1) is $D_{\mathbf{r}}-$homogeneous of degree $\nu = 1$, under some stability assumptions of the system's origin, the existence of an $\mathbf{r}$-homogeneous LF is guaranteed[20]. The following theorem states such a result, but with simpler assumptions and proof than those in[20]. We use the definition of *robust global asymptotic stability* as given in[23], see also Appendix C. The origin of (1) is said to be robustly globally asymptotically stable (robustly GAS) if its associated difference inclusion is robustly strongly GAS (see Appendix C).

Theorem 5. Suppose that (1) is $D_{\mathbf{r}}-$homogeneous of degree $\nu = 1$, $x = 0$ is the unique equilibrium point. If $x = 0$ is locally asymptotically stable, then $x = 0$ is globally asymptotically stable. Moreover, in any of the following cases: 1) f is continuous; 2) f is discontinuous, and $x = 0$ is robustly GAS; there exists a differentiable LF $V : \mathbb{R}^n \to \mathbb{R}$ for (1) which is $\mathbf{r}-$homogeneous of degree m for any $m > \max_i(r_i)$.

Proof. The homogeneity and stability assumptions imply that $x = 0$ is globally asymptotically stable, the proof is the same as in[19] Claim 1, but considering the discontinuous case.

The existence of a differentiable and radially unbounded LF $V_1 : \mathbb{R}^n \to \mathbb{R}$ for (1) is guaranteed in[25] Theorem 1 for f continuous. For the discontinuous case, in[23] Theorems 10, 12 and 14, the existence of V_1 is guaranteed for the difference inclusion associated to (1), see also Appendix C. To construct an $\mathbf{r}-$homogeneous Lyapunov function we use the technique provided in[4]. The construction is made for the associated differential inclusion $x(k+1) \in F(x(k))$, the case of f continuous is a particular one. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$\psi(s) = \begin{cases} 0, & s \in (-\infty, 1], \\ 1, & s \in [2, \infty), \end{cases}$$

[2] For $\alpha \in \mathbb{R}_{>0}$, $\alpha \in (0,1]$, a function $V : \mathbb{R}^n \to \mathbb{R}$ is called $\alpha-$Hölder continuous in the set $I \subset \mathbb{R}^n$, if there exists $L_I \in \mathbb{R}_{>0}$ such that $|V(x) - V(y)| \leq L_I|x-y|^\alpha$ for all $x, y \in I$, see e.g.[24].

and is strictly increasing in the interval $(1,2)$. Consider the function $V:\mathbb{R}^n\to\mathbb{R}$ given by

$$V(x)=\int_0^\infty \frac{1}{\epsilon^{m+1}}\psi\left(V_1(\Lambda_\epsilon^{\mathbf{r}}x)\right)\,\mathrm{d}\epsilon\,. \tag{6}$$

Homogeneity, positive definiteness and smoothness properties of V are proven exactly the same as in[4]. We verify now that $\bar{\Delta}V(x):=\max_{y\in F(x)}V(y)-V(x)<0$ for all $x\neq 0$. First note that, since F is $\boldsymbol{D}_{\mathbf{r}}-$homogeneous of degree $\nu=1$, and ψ is non-decreasing,

$$\max_{y\in F(x)}V(y)\le\int_0^\infty\frac{1}{\epsilon^{m+1}}\psi\left(\max_{y\in F(x)}V_1(\Lambda_\epsilon^{\mathbf{r}}y)\right)\,\mathrm{d}\epsilon=\int_0^\infty\frac{1}{\epsilon^{m+1}}\psi\left(\max_{w\in F(\Lambda_\epsilon^{\mathbf{r}}x)}V_1(w)\right)\,\mathrm{d}\epsilon\,,$$

where $w=\Lambda_\epsilon^{\mathbf{r}}y$. Now, since $\max_{y\in F(x)}V_1(y)$ is strictly less than $V_1(x)$ for all $x\neq 0$, and ψ is strictly increasing in $(1,2)$, we have that

$$\bar{\Delta}V(x)\le\int_0^\infty\frac{1}{\epsilon^{m+1}}\left[\psi\left(\max_{w\in F(\Lambda_\epsilon^{\mathbf{r}}x)}V_1(w)\right)-\psi\left(V_1(\Lambda_\epsilon^{\mathbf{r}}x)\right)\right]\,\mathrm{d}\epsilon<0\,,$$

for all $x\neq 0$. In particular, we have that (along the solutions of (1)) $\Delta V(x)<0$, hence, V is a LF for (1). Now, since V is continuous and locally bounded, the unit homogeneous sphere S is compact, and $F(x)$ is compact for all $x\in\mathbb{R}^n$, then the set $A=\{z\in\mathbb{R}\,:\,z=V(w)-V(y),\ y\in S,\ w\in F(y)\}$ is compact. Moreover, $z<0$ for all $z\in A$, and $\{z\in\mathbb{R}\,:\,z=\Delta V(y),\ y\in S\}\subset A$. Hence, $\inf_{y\in S}(-\Delta V(y))>0$. Therefore, according to Lemma 6, there exists $\chi\in\mathbb{R}_{>0}$ such that $\chi\|x\|_{\mathbf{r}}^m\le-\Delta V(x)$ for all $x\in\mathbb{R}^n$. □

4 | CONVERGENCE RATES OF $\boldsymbol{D}_{\mathbf{r}}$−HOMOGENEOUS SYSTEMS

In this section we show that, as in the CT case, the $\boldsymbol{D}_{\mathbf{r}}-$homogeneity degree of a DT system determines the convergence rate of its trajectories. For the definition of Hyper-exponential stability see[26] Definitions 12.5 and 12.6.

Theorem 6. Let (1) be $\boldsymbol{D}_{\mathbf{r}}$-homogeneous of degree ν.

a) If $\nu=1$ and either: 1) f is continuous, and $x=0$ is asymptotically stable or; 2) f is discontinuous, and $x=0$ is robustly GAS; then there exist constants $\mu_1,\mu_2\in\mathbb{R}_{>0}$ such that, for any $x(0)\in\mathbb{R}^n$,

$$\|x(k)\|_{\mathbf{r}}\le\mu_1\|x(0)\|_{\mathbf{r}}\exp(g(k))\,,\quad g(k)=-\mu_2k\,,\quad\forall\,k\in\mathbb{Z}_{\ge0}\,. \tag{7}$$

b) Let V and α be as in Theorem 2. If $\nu>1$, then $x=0$ is locally uniformly hyper-exponentially stable in $\mathcal{I}(V,\alpha)$, i.e. there exist constants $\mu_1,\mu_2,\mu_3\in\mathbb{R}_{>0}$ such that, for any $x(0)\in\mathcal{I}(V,\alpha)$,

$$\|x(k)\|_{\mathbf{r}}\le\mu_1\|x(0)\|_{\mathbf{r}}\exp(g(k))\,,\quad g(k)=-\mu_2\exp(\mu_3k)\,,\quad\forall\,k\in\mathbb{Z}_{\ge0}\,. \tag{8}$$

c) Let V and $\bar{\alpha}$ be as in Theorem 3. If $\nu\in(0,1)$, then there exist constants $\mu_1,\mu_2,\mu_3\in\mathbb{R}_{>0}$ such that, for any $x(0)\in\mathcal{E}(V,\bar{\alpha})$,

$$\|x(k)\|_{\mathbf{r}}\le\mu_1\exp(g(k))\,,\quad g(k)=\ln(\mu_2\|x(0)\|_{\mathbf{r}})\exp(-\mu_3k)\,, \tag{9}$$

for all $k\in\mathbb{Z}_{\ge0}$ such that $x(k)\in\mathcal{E}(V,\bar{\alpha})$.

According to[26] Definition 12.6, the convergence rate for $\nu=1$ is exponential or hyper-exponential of degree zero, and the convergence rate for $\nu>1$ is hyper-exponential of degree one. Note that, in any case of Theorem 6, g is a strictly decreasing function of k. However, in (9), g is bounded from below by zero, in accordance to the practical stability property.

Proof. According to Lemma 5, if V is a continuous, positive definite, $\mathbf{r}-$homogeneous function of degree $m\in\mathbb{R}_{>0}$, then there exist constants $\underline{\theta},\overline{\theta}\in\mathbb{R}_{>0}$ such that

$$\underline{\theta}\|x\|_{\mathbf{r}}^m\le V(x)\le\overline{\theta}\|x\|_{\mathbf{r}}^m\,. \tag{10}$$

a) From Theorem 5 we know that there exists a LF V for (1) that is $\mathbf{r}-$homogeneous of some degree m, and from its proof, that there exists $\chi\in\mathbb{R}_{>0}$ such that the function $-\Delta V$ given by $-\Delta V(x)=-V(f(x))+V(x)$ satisfies

$$\chi\|x\|_{\mathbf{r}}^m\le-\Delta V(x)\,. \tag{11}$$

Thus, from (10) and (11), it is easy to see that

$$V(f(x)) \leq \left(1 - \chi/\overline{\theta}\right) V(x)\,, \quad 0 < 1 - \chi/\overline{\theta} < 1\,. \tag{12}$$

Therefore, by defining $\lambda = 1 - \chi/\overline{\theta}$, recalling the dependence of x on k, and using recursively (12), we obtain

$$V(x(k)) \leq \lambda^k V(x(0)) = V(x(0)) \exp\left(-\ln(1/\lambda)k\right)\,. \tag{13}$$

Hence, by using (10) in (13) we have that

$$\underline{\theta}\|x(k)\|_{\mathbf{r}}^m \leq \overline{\theta}\|x(0)\|_{\mathbf{r}}^m \exp\left(-\ln(1/\lambda)k\right)\,,$$

and from this inequality we obtain (7) with $\mu_1 = (\overline{\theta}/\underline{\theta})^{\frac{1}{m}}$, $\mu_2 = \ln(1/\lambda)/m$.

b) As it has been established in the proof of Theorem 2 (see (3)), $\Delta V(x) \leq \overline{\gamma} V^{\nu}(x) - V(x)$. Hence, $V(x(k+1)) \leq \overline{\gamma} V^{\nu}(x(k))$, and from a recursive use of this inequality we obtain

$$V(x(k)) \leq V^{\nu^k}(x(0)) \prod_{i=0}^{k-1} \overline{\gamma}^{\nu^i} = V(x(0)) V^{\nu^k - 1}(x(0)) \overline{\gamma}^{\frac{\nu^k-1}{\nu-1}}\,. \tag{14}$$

Recall that the domain is restricted to $\mathcal{I}(V, \alpha)$ and $\alpha = \left(\frac{1-\varepsilon}{\overline{\gamma}}\right)^{\frac{1}{\nu-1}}$, for any $\varepsilon \in (0,1)$. Thus, $V(x(k)) < \left(\frac{1-\varepsilon}{\overline{\gamma}}\right)^{\frac{1}{\nu-1}}$, or equivalently, $V(x(k))\overline{\gamma}^{\frac{1}{\nu-1}} < (1-\varepsilon)^{\frac{1}{\nu-1}}$. In particular, $\overline{\gamma}^{\frac{1}{\nu-1}} V(x(0) < (1-\varepsilon)^{\frac{1}{\nu-1}}$, and from (14) we obtain

$$V(x(k)) \leq V(x(0)) \left((1-\varepsilon)^{\frac{1}{\nu-1}}\right)^{\nu^k-1} \leq V(x(0)) \exp\left(-\ln\left((1-\varepsilon)^{\frac{-1}{\nu-1}}\right)\left(\mathrm{e}^{\ln(\nu)k} - 1\right)\right)\,. \tag{15}$$

By using (10), we obtain from (15)

$$\underline{\theta}\|x(k)\|_{\mathbf{r}}^m \leq \overline{\theta}\|x(0)\|_{\mathbf{r}}^m \exp\left(-\ln\left((1-\varepsilon)^{\frac{-1}{\nu-1}}\right)\left(\mathrm{e}^{\ln(\nu)k} - 1\right)\right)\,,$$

and from this inequality, (8) is obtained with $\mu_1 = (\overline{\theta}/\underline{\theta})^{\frac{1}{m}} \mathrm{e}^{\mu_2}$, $\mu_2 = \ln\left((1-\varepsilon)^{\frac{-1}{\nu-1}}\right)/m$, and $\mu_3 = \ln(\nu)$.

c) Analogously to b) we have that $V(x(k)) \leq V^{\nu^k}(x(0))\overline{\gamma}^{\frac{\nu^k-1}{\nu-1}}$. Hence,

$$V(x(k)) \leq \left(\frac{V(x(0))}{\overline{\gamma}^{\frac{1}{1-\nu}}}\right)^{\nu^k} \overline{\gamma}^{\frac{1}{1-\nu}} \leq \overline{\gamma}^{\frac{1}{1-\nu}} \exp\left(\ln\left[\frac{V(x(0))}{\overline{\gamma}^{\frac{1}{1-\nu}}}\right] \mathrm{e}^{-\ln(1/\nu)k}\right)\,. \tag{16}$$

By using (10), we obtain from (16)

$$\underline{\theta}\|x(k)\|_{\mathbf{r}}^m \leq \overline{\gamma}^{\frac{1}{1-\nu}} \exp\left(\ln\left[\frac{\overline{\theta}\|x(0)\|_{\mathbf{r}}^m}{\overline{\gamma}^{\frac{1}{1-\nu}}}\right] \mathrm{e}^{-\ln(1/\nu)k}\right)\,,$$

and from this inequality, (9) is obtained with $\mu_1 = \left(\overline{\gamma}^{\frac{1}{1-\nu}}/\underline{\theta}\right)^{\frac{1}{m}}$, $\mu_3 = \ln(1/\nu)$, and $\mu_2 = \left(\frac{\overline{\theta}}{\overline{\gamma}^{\frac{1}{1-\nu}}}\right)^{\frac{1}{m}}$.

□

5 | EXAMPLES

In this section we illustrate through some simulations the results obtained in previous sections. Consider the DT system

$$\begin{aligned} x_i(k+1) &= x_{i+1}(k) + \delta_i(k)\,,\; i = 1, \dots, n-1\,, \\ x_n(k+1) &= u + \delta_n(k)\,, \end{aligned} \tag{17}$$

where $\delta(k) = [\delta_1(k), \dots, \delta_n(k)]^\top \in \mathbb{R}^n$ is a vector of disturbances, and u is a feedback controller given by

$$u = -\sum_{i=1}^{n} a_i \lceil x_i(k) \rfloor^{\rho_i}\,, \quad \rho_i = \nu^{n-i+1}\,, \quad \nu, a_i \in \mathbb{R}_{>0}\,. \tag{18}$$

If $\delta(k) = 0$ for all $k \in \mathbb{Z}$, then: the transition map of the closed loop (17), (18) is $\boldsymbol{D}_{\mathbf{r}}$-homogeneous of degree ν with $\mathbf{r} = [r_1, r_1\nu, \dots, r_1\nu^{n-1}]^\top$ for any $r_1 \in \mathbb{R}_{>0}$; according to Theorem 2 and Theorem 3, the origin of (17), (18), is locally asymptotically stable for $\nu > 1$, and globally practically stable for $\nu \in (0,1)$, for any $a_i \in \mathbb{R}_{>0}$; when $\nu = 1$, the system is linear and the parameters a_i must be correctly designed to guarantee asymptotic stability of the origin. If $|\delta(k)| \leq \Delta$ for all $k \in \mathbb{Z}$, and

FIGURE 1 States of (19) for $v = 2/3$, disturbed case.

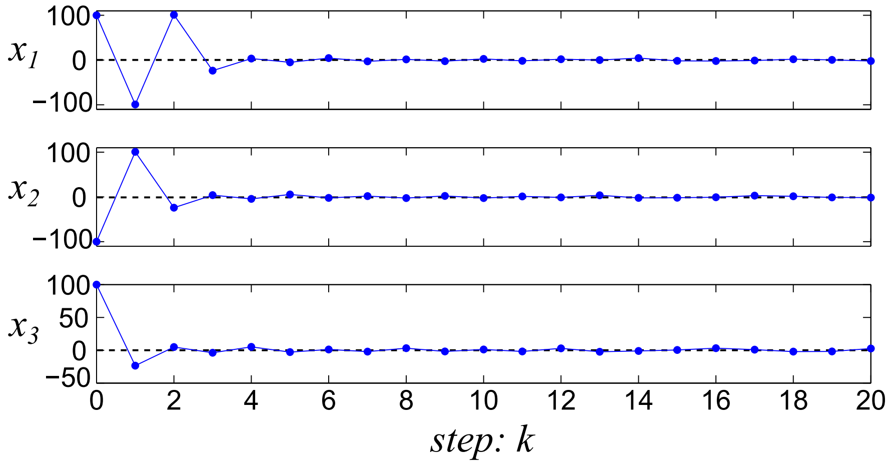
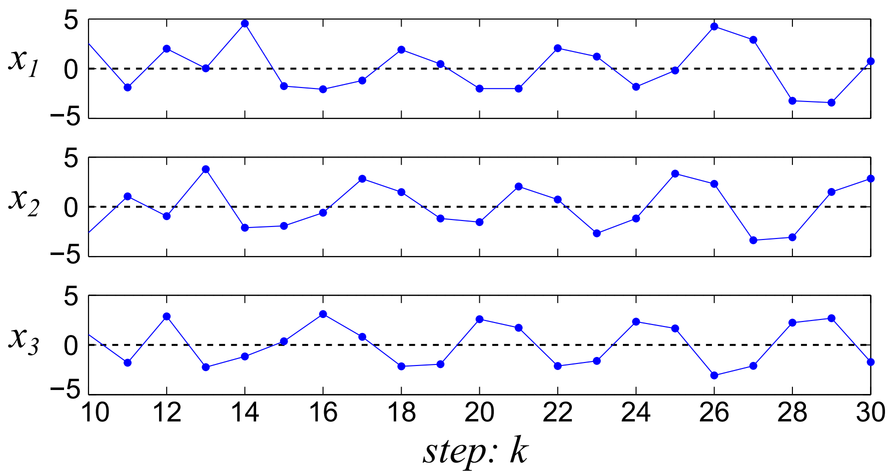


FIGURE 2 Zoom of the states of (19) for $v = 2/3$, disturbed case.



$v \in (0, 1)$, then the system's origin is practically stable for any finite $\Delta \in \mathbb{R}_{\geq 0}$. In[21], some simulations of (17) has been shown for the undisturbed case.

Example 1. In the case of $n = 3$, the closed loop (17), (18) is

$$\begin{aligned} x_i(k+1) &= x_{i+1}(k) + \delta_i(k), \quad i = 1, 2, \\ x_3(k+1) &= -\sum_{i=1}^{3} a_i \lceil x_i(k) \rfloor^{v^{4-i}} + \delta_3(k). \end{aligned} \tag{19}$$

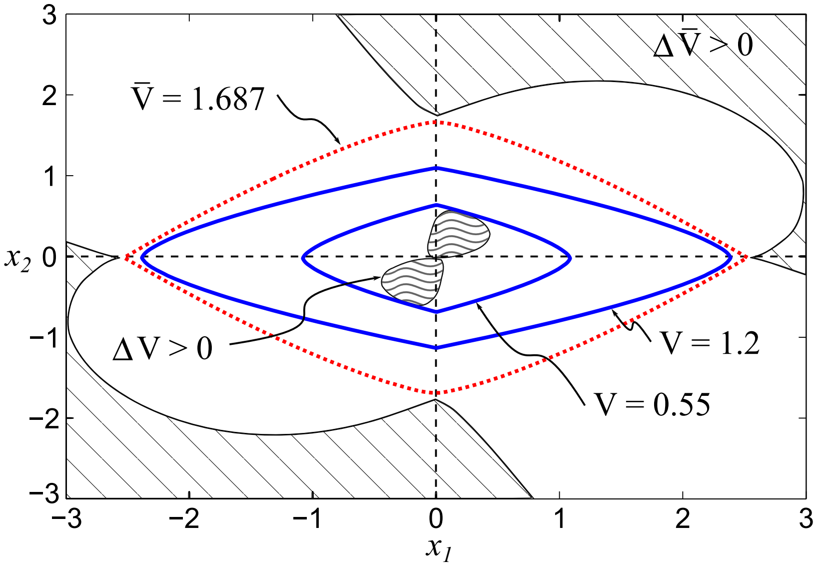
We consider $a_1 = a_2 = a_3 = 4/3$ in (19). Observe that in the linear case ($v = 1$) these gains would render the origin unstable.

To illustrate the robustness property stated in Theorem 4, let us consider (19). If $v = 2/3$, then the origin of (19) is practically stable in the nominal case. With the bounded disturbances $\delta_1(k) = \cos(k/2)$, $\delta_2(k) = \cos(k)$, and $\delta_3(k) = \cos(3k/2)$, Fig. 1 and Fig. 2 show that the origin of (19) preserves the practical stability property despite the disturbances. The initial conditions $x_1(0) = 100$, $x_2(0) = -100$, $x_3(0) = 100$ were used for the simulation.

Example 2. Now we illustrate an application of $D_{\mathbf{r}}-$homogeneity for global asymptotic stabilization by means of a switched controller. A similar example is shown in[21], nonetheless, here we use LFs to design the switching strategy. Consider again (17) with $n = 2$ and each $\delta_i(k) = 0$, for all $k \in \mathbb{Z}$, i.e.

$$x_1(k+1) = x_2(k), \quad x_2(k+1) = -a_1 \lceil x_1(k) \rfloor^{v^2} - a_2 \lceil x_2(k) \rfloor^{v}. \tag{20}$$

FIGURE 3 Level sets of V and $\bar{V}$.



Since the proposed strategy will be compared with the linear case ($\nu = 1$), we select $a_1 = a_2 = 1/3$ for both control schemes. The main idea is to use one $D_{\mathbf{r}}$−homogeneity degree to bring the system trajectories to a vicinity of the origin, and then to change the $D_{\mathbf{r}}$−homogeneity degree to drive the systems trajectories to the origin. Thus, define the following switching strategy:

$$\text{Set } \nu = \begin{cases} 2/3 & \text{if } \ V(x(k)) \geq \vartheta\,, \\ 4/3 & \text{if } \ V(x(k)) < \vartheta\,, \end{cases}$$

for some function $V : \mathbb{R}^2 \to \mathbb{R}$ and some $\vartheta \in \mathbb{R}_{>0}$. The design of V and ϑ is Lyapunov based, and it consists in using the estimations of the attraction domains and the region of practical stability from Theorems 2 and 3.

For $\nu = 2/3$ we chose the Lyapunov-like function $V : \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(x) = \tfrac{1}{2}|x_1| + |x_2|^{\frac{3}{2}}\,. \tag{21}$$

Note that it is $\mathbf{r}$−homogeneous of degree $m = 3$ for $\mathbf{r} = [3, 2]^\top$, and (we are omitting the dependence on k)

$$\begin{aligned} \Delta V(x) &= \tfrac{1}{2}|x_2| + \left|\tfrac{1}{3}\lceil x_1\rfloor^{\nu^2} + \tfrac{1}{3}\lceil x_2\rfloor^{\nu}\right|^{\frac{3}{2}} - \tfrac{1}{2}|x_1| - |x_2|^{\frac{3}{2}}\,, \\ &\leq -\left(\tfrac{1}{2}|x_1|^{\frac{1}{3}} - \sqrt{2}(1/3)^{\frac{3}{2}}\right)|x_1|^{\frac{2}{3}} - \left(|x_2|^{\frac{1}{2}} - \tfrac{1}{2} - \sqrt{2}(1/3)^{\frac{3}{2}}\right)|x_2|\,. \end{aligned}$$

Hence, $\Delta V(x(k)) < 0$ for all $x \in \mathbb{R}^2$ such that $|x_1| > (2/3)^{\frac{9}{2}}$ and $|x_2| > \frac{1}{4}\left(1 + (2/3)^{\frac{3}{2}}\right)^2$. Therefore, $\Delta V(x(k)) < 0$ for all $x \in \mathbb{R}^2$ such that $V(x) > \frac{1}{2}(2/3)^{\frac{9}{2}} + \frac{1}{8}\left(1 + (2/3)^{\frac{3}{2}}\right)^3 \approx 0.541$.

For $\nu = 4/3$ we chose the Lyapunov function $\bar{V} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\bar{V}(x) = \tfrac{1}{2}|x_1|^{\frac{4}{3}} + |x_2|\,. \tag{22}$$

Note that it is $\mathbf{r}$−homogeneous of degree $m = 4$ for $\mathbf{r} = [3, 2]^\top$, and

$$\begin{aligned} \Delta \bar{V}(x) &= \tfrac{1}{2}|x_2|^{\frac{4}{3}} + \tfrac{1}{3}\left|\lceil x_1\rfloor^{\nu^2} + \lceil x_2\rfloor^{\nu}\right| - \tfrac{1}{2}|x_1|^{\frac{4}{3}} - |x_2|\,, \\ &\leq -\left(\tfrac{1}{2} - \tfrac{1}{3}|x_1|^{\frac{4}{9}}\right)|x_1|^{\frac{4}{3}} - \left(1 - \tfrac{5}{6}|x_2|^{\frac{1}{3}}\right)|x_2|\,. \end{aligned}$$

Hence, $\Delta\bar{V}(x(k)) < 0$ for all $x \in \mathbb{R}^2$ such that $|x_1| < (3/2)^{\frac{9}{4}}$ and $|x_2| < (6/5)^3$. To find an estimate of the attraction region, it is enough to analyse $\bar{V}$ on the coordinate axes by considering the set where $\Delta\bar{V}(x) < 0$. Therefore, $\Delta\bar{V}(x(k)) < 0$ for all $x \in \mathbb{R}^2$ such that $V(x) < \min\left(\frac{1}{2}(3/2)^3,\ (6/5)^3\right) \approx 1.687$. In Fig. 3, some level curves for V and $\bar{V}$ can be appreciated. Hence, for the switching strategy, we can choose V as in (21) and any $\vartheta \in [0.55,\ 1.2]$. The simulation is performed with the initial conditions

FIGURE 4 States of (19) (undisturbed) with Linear ($\nu = 1$) and Switched schemes.

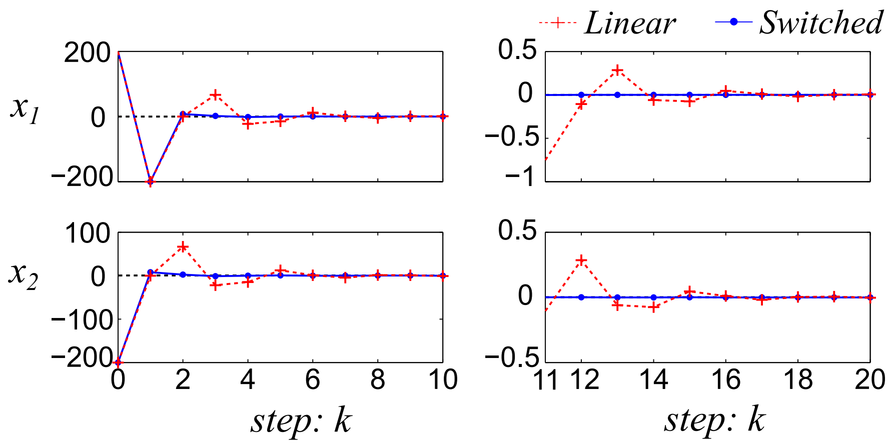
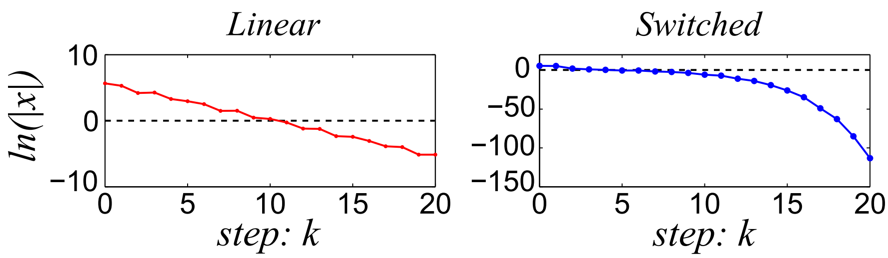


FIGURE 5 Comparison of convergence rates of Linear and Switched schemes.



$x_1(0) = 200$, $x_2(0) = -200$. Fig. 4 shows the responses of the linear and the switched schemes. Note that, as expected, the switched scheme is able to drive the system's trajectories to a vicinity of the origin faster than the linear controller.

Fig. 5 shows the convergence rate of both schemes, the hyper-exponential rate of the switched scheme is confirmed.

Now we consider (20) in presence of disturbances, i.e.

$$x_1(k+1) = x_2(k), \quad x_2(k+1) = -a_1\lceil x_1(k)\rfloor^{\nu^2} - a_2\lceil x_2(k)\rfloor^{\nu} + \delta(k),$$

where $\delta(k) = 1 + 2\cos(3k/2)$. Fig. 6 shows the responses of the linear and the switched schemes. Note that, for both control schemes the system's origin is practically stable. However, the trajectories with the switched scheme converge faster to the practical stability region, see Fig. 7.

6 | CONCLUSION

In this paper the concept of $D_{\mathbf{r}}-$homogeneity was introduced. This is a notion of homogeneity for nonlinear DT systems, which generalizes the previously introduced frameworks of [19] and [20]. $D_{\mathbf{r}}-$homogeneity allows the trajectories of $D_{\mathbf{r}}-$homogeneous DT systems to be scaled, as in the CT case. Contrarily the latter, in the DT this does not lead to a global stability/instability of the system, but (for $\nu \neq 1$) it simplifies greatly the local stability analysis (and convergence rate) of the system, since it becomes a function that depends only on the $D_{\mathbf{r}}-$homogeneity degree. In addition, it was shown that the selection of LFs for $D_{\mathbf{r}}-$homogeneous DT systems does not represent a problem, nonetheless, its adjustment allows the domain of attraction (or repulsion) to be better evaluated. The theoretical findings are illustrated by a nonlinear stabilization control design with the results of simulations confirming efficiency of the concept.

FIGURE 6 States of (19) (disturbed) with Linear ($\nu = 1$) and Switched schemes.

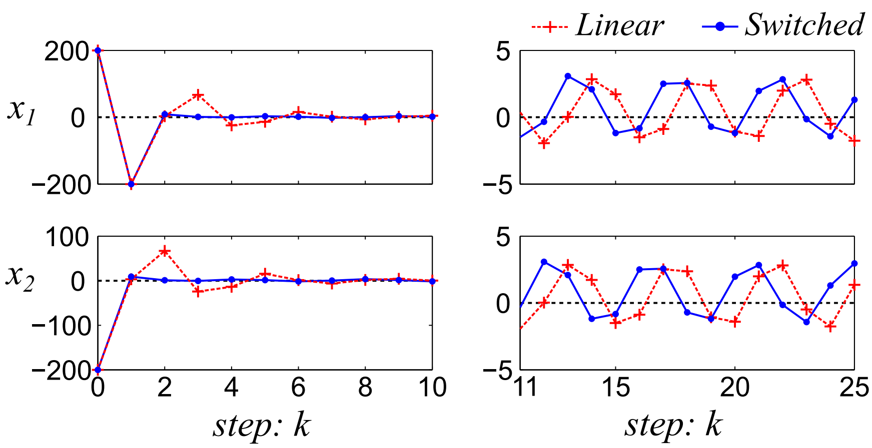
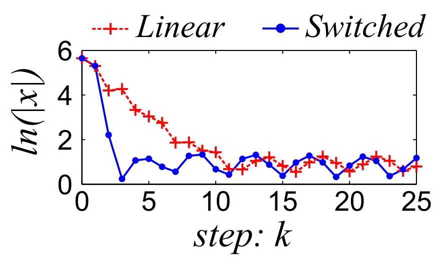


FIGURE 7 Comparison of convergence rates of Linear and Switched schemes (disturbed case).



The directions of future research include: extension of the definitions of $D_{\mathbf{r}}-$homogeneity to generalized dilation operators; analysis of additional robustness properties of $D_{\mathbf{r}}-$homogeneous systems; investigation of general schemes for control for different $D_{\mathbf{r}}-$homogeneity degrees; introduction of local approximations and limit homogeneity; and studying the possibility to design $D_{\mathbf{r}}-$homogeneous observers.

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□

APPENDIX

A STABILITY DEFINITIONS

The following definitions were taken from [14].

Definition 3. Consider (1) with an equilibrium point at the origin, i.e. $f(0) = 0$. Let $F(k; x_0)$ be the system's solution defined for all $k \in \mathbb{Z}_{\geq 0}$, for the initial condition x_0. The system's origin is:

- stable if for each $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta(\varepsilon) \in \mathbb{R}_{>0}$ such that for any solution F, the inequality $|x_0| < \delta$ implies that $F(k; x_0)$ is defined for all $k \in \mathbb{Z}_{\geq 0}$, and $|F(k; x_0)| < \varepsilon$ for all $k \geq 0$;
- asymptotically stable if it is stable and δ can be chosen such that $|x_0| < \delta$ implies $|F(k; x_0)| \to 0$ as $k \to \infty$;
- unstable if it is not stable.

Definition 4. Let $F(k; x_0)$ (defined for all $k \in \mathbb{Z}_{\geq 0}$) denote the solution of (1) for the initial condition x_0. The origin of (1) is *globally practically stable*, if there exists a neighbourhood $\mathcal{S} \subset \mathbb{R}^n$ of $x = 0$, and for every $\iota \in \mathbb{R}_{>0}$, there is $k_1 = k_1(\iota) \in \mathbb{Z}_{\geq 0}$ such that: $|x_0| \leq \iota \Rightarrow F(k, x_0) \in \mathcal{S}\ \forall k \geq k_1$.

B HOMOGENEOUS FUNCTIONS

We state some useful properties of $\mathbf{r}-$homogeneous functions.

Definition 5 (See e.g.[9]). Given a vector of weights $\mathbf{r}$, a $\mathbf{r}-$homogeneous norm is defined as a function from $\mathbb{R}^n$ to $\mathbb{R}$, and given by $\|x\|_{\mathbf{r},p} = \left(\sum_{i=1}^n |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}}$, $\forall x \in \mathbb{R}^n$, for any $p \geq 1$. The set $S_1 = \{x \in \mathbb{R}^n \ :\ \|x\|_{\mathbf{r},p} = 1\}$ is the corresponding $\mathbf{r}-$homogeneous unit sphere.

Note that any $\mathbf{r}-$homogeneous norm is a $\mathbf{r}-$homogeneous function of degree $m = 1$. Since, for a given $\mathbf{r}$, the $\mathbf{r}-$homogeneous norms are equivalent[22], they are usually denoted as $\|\cdot\|_{\mathbf{r}}$, without the specification of p.

Lemma 5 ([10]). Suppose that $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous $\mathbf{r}$-homogeneous functions of degrees $m_1, m_2 \in \mathbb{R}_{>0}$, respectively, and V_1 is positive definite. Then,

$$\underline{\gamma} V_1^{\frac{m_2}{m_1}}(x) \leq V_2(x) \leq \overline{\gamma} V_1^{\frac{m_2}{m_1}}(x),$$

for every $x \in \mathbb{R}^n$, where $\underline{\gamma} = \min_{x \in E} V_2(x)$, and $\overline{\gamma} = \max_{x \in E} V_2(x)$, with $E = \{x \in \mathbb{R}^3 \ :\ V_1(x) = 1\}$.

Remark 3. In Lemma 5, if V_2 is positive semi-definite, then $\underline{\gamma} \in \mathbb{R}_{\geq 0}$ and $\overline{\gamma} \in \mathbb{R}_{>0}$. Nonetheless, if V_2 is positive definite, then $\underline{\gamma}, \overline{\gamma} \in \mathbb{R}_{>0}$.

An analogous result can be stated for discontinuous functions.

Lemma 6. Suppose that the function $V : \mathbb{R}^n \to \mathbb{R}$ is well defined and locally bounded for all $x \in \mathbb{R}^n$, it is $\mathbf{r}-$homogeneous of degree $m \in \mathbb{R}_{>0}$, and it is positive semi-definite. Then:
1) there exists $\overline{\gamma} \in \mathbb{R}_{>0}$ such that

$$V(x) \leq \overline{\gamma}\|x\|_{\mathbf{r}}^m, \quad \forall\, x \in \mathbb{R}^n;$$

2) if, additionally, there exists $\underline{\gamma} \in \mathbb{R}_{>0}$ such that $\inf_{y\in S} V(y) \geq \underline{\gamma}$, $S = \{x \in \mathbb{R}^n \ : \ \|x\|_{\mathbf{r}} = 1\}$ then

$$V(x) \geq \underline{\gamma}\|x\|_{\mathbf{r}}^m, \quad \forall\, x \in \mathbb{R}^n.$$

Proof. 1) Since V is locally bounded, we can assure that there exists $\delta \in \mathbb{R}_{>0}$ such that $V(y) \leq \delta$ for all $y \in S$, indeed we can choose $\delta = \sup_{y\in S} V(y)$. Note that, if for any $x \neq 0$ we set $\epsilon = \|x\|_{\mathbf{r}}^{-1}$, then $y = \Lambda_\epsilon^{\mathbf{r}} x \in S$. Thus, $\delta \geq V(y) = V(\Lambda_\epsilon^{\mathbf{r}} x) = (\|x\|_{\mathbf{r}}^{-1})^m V(x)$, then $V(x) \leq \overline{\gamma}\|x\|_{\mathbf{r}}^m$ for any $\overline{\gamma} \geq \delta$. 2) The proof of this point is analogous to the previous one. □

A consequence of lemmas 5 and 6 is the following lemma.

Lemma 7. If $V : \mathbb{R}^n \to \mathbb{R}$ is a continuous function that is positive definite and $\mathbf{r}-$homogeneous of degree $m \in \mathbb{R}_{>0}$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map that is well defined and locally bounded for all $x \in \mathbb{R}^n$, and is $D_{\mathbf{r}}-$homogeneous of degree $\nu \in \mathbb{R}_{>0}$, then: 1) there exists $\overline{\gamma} \in \mathbb{R}_{>0}$ such that

$$(V \circ f)(x) \leq \overline{\gamma} V^\nu(x), \quad \forall\, x \in \mathbb{R}^n;$$

2) if additionally, f satisfies **A1**, then there exists $\underline{\gamma} \in \mathbb{R}_{>0}$ such that

$$(V \circ f)(x) \geq \underline{\gamma} V^\nu(x), \quad \forall\, x \in \mathbb{R}^n.$$

Proof. First note that since f is $D_{\mathbf{r}}-$homogeneous, then $f(0) = 0$, therefore, $(V \circ f)(0) = 0$. From Lemma 1 we know that $V \circ f$ is positive semi-definite, $\mathbf{r}-$homogeneous of degree $\bar{m} = m\nu$, and locally bounded. 1) From Lemma 6, $(V \circ f)(x) \leq \delta\|x\|_{\mathbf{r}}^{m\nu}$, and from Lemma 5, there exists $\tilde{\gamma} \in \mathbb{R}_{>0}$ such that $\|x\|_{\mathbf{r}}^{m\nu}\tilde{\gamma} \leq V^\nu(x)$. Hence, we get the result with $\overline{\gamma} = \delta/\tilde{\gamma}$. 2) By $D_{\mathbf{r}}-$homogeneity of f, assumption **A1**, and continuity of V, we have that there exists $\delta \in \mathbb{R}_{>0}$ such that $\inf_{y\in S}(V \circ f)(y) \geq \delta$, where $S = \{x \in \mathbb{R}^n \ : \ \|x\|_{\mathbf{r}} = 1\}$. The results follows from Lemmas 6 and 5. □

Lemma 8. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite $\mathbf{r}-$homogeneous function of degree m. Suppose that V is $\alpha-$Hölder continuous for any compact set of $\mathbb{R}^n$. Define $\rho = m - \alpha \min_i\{r_i\}$. For any finite $Y \in \mathbb{R}_{\geq 0}$ there exists $L_Y \in \mathbb{R}_{>0}$ such that, for all $x \in \mathbb{R}^n$ and all $y \in \{y \in \mathbb{R}^n \ : \ |y| \leq Y\}$

$$|V(x+y) - V(x)| \leq \begin{cases} L_Y|y|^\alpha & \text{if } \ \|x\|_{\mathbf{r}} \leq 1, \\ L_Y\|x\|_{\mathbf{r}}^\rho|y|^\alpha & \text{if } \ \|x\|_{\mathbf{r}} \geq 1. \end{cases}$$

Proof. Define $\bar{Y} = \max_{y\in\{y\in\mathbb{R}^n : |y|=Y\}} \|y\|_{\mathbf{r}}$, note that $\{y \in \mathbb{R}^n \ : \ |y| \leq Y\} \subset \{y \in \mathbb{R}^n \ : \ \|y\|_{\mathbf{r}} \leq \bar{Y}\}$. If $\|x\|_{\mathbf{r}} \leq 1$, then $x, y \in \{w \in \mathbb{R}^n \ : \ \|w\|_{\mathbf{r}} \leq \max(1, \bar{Y})\}$, since this set is compact and V is $\alpha-$Hölder continuous, then there exists $L_Y \in \mathbb{R}_{>0}$ such that $|V(x+y) - V(x)| \leq L_Y|y|^\alpha$.

Now, suppose that $\|x\|_{\mathbf{r}} \geq 1$. Consider the change of coordinates $z = \Lambda_\epsilon^{\mathbf{r}} x$, and define $\bar{y} = \Lambda_\epsilon^{\mathbf{r}} y$ with $\epsilon = 1/\|x\|_{\mathbf{r}}$. Note that $\|\bar{y}\|_{\mathbf{r}} = \|y\|_{\mathbf{r}}/\|x\|_{\mathbf{r}}$, therefore $0 \leq \|\bar{y}\|_{\mathbf{r}} \leq \|y\|_{\mathbf{r}}$. Hence, by their construction, $z, \bar{y} \in \{w \in \mathbb{R}^n \ : \ \|w\|_{\mathbf{r}} \leq \max(1, \bar{Y})\}$, for all $x \in \mathbb{R}^n$. Thus, $|V(x+y) - V(x)| = |V\left((\Lambda_\epsilon^{\mathbf{r}})^{-1}(z+\bar{y})\right) - V((\Lambda_\epsilon^{\mathbf{r}})^{-1}z)| = \|x\|_{\mathbf{r}}^m|V(z+\bar{y}) - V(z)| \leq \|x\|_{\mathbf{r}}^m L_Y|\bar{y}|^\alpha$. On the other hand, $|\bar{y}| = |\Lambda_\epsilon^{\mathbf{r}} y| = \left(\sum_{i=1}^n \frac{y_i^2}{\|x\|_{\mathbf{r}}^{2r_i}}\right)^{\frac{1}{2}} \leq \left(\|x\|_{\mathbf{r}}^{-2\min_i\{r_i\}} \sum_{i=1}^n y_i^2\right)^{\frac{1}{2}}$, this completes the proof. □

C DIFFERENCE INCLUSIONS

Consider (1) and its associated difference inclusion[23]

$$x(k+1) \in F(x(k)), \quad x(k) \in \mathbb{R}^n, \tag{C1}$$

where the set-valued map $F(x) \subset \mathbb{R}^n$ is given by $F(x) = \bigcap_{\rho\in\mathbb{R}_{>0}} \mathrm{cl}\{f(B(x,\rho)\}$, where $B(x,\rho)$ is an open ball in $\mathbb{R}^n$ centered at x with radius ρ, and $\mathrm{cl}\{A\}$ denotes the closure of the set A. The set-valued map F is nonempty, compact, and upper-semicontinuous for all $x \in \mathbb{R}^n$[23].

Definition 6 ([23],[27])**.** The origin of (C1) is strongly GAS if there exists a class-KL function β such that, for every $x_0 \in \mathbb{R}^n$, all the solutions Φ with initial condition x_0 satisfy $|\Phi(k; x_0)| \leq \beta(|x_0|, k)$ for all $k \in \mathbb{Z}_{\geq 0}$.

Definition 7 ([23],[27])**.** The origin of (C1) is robustly strongly GAS if there exists a continuous and positive definite function $\delta : \mathbb{R}^n \to \mathbb{R}$ such that the origin of $x(k+1) \in F_\delta(x(k))$ is strongly GAS with $F_\delta(x) = \{y \in \mathbb{R}^n \;:\; y \in F(w) + \mathrm{cl}\{B(0, \delta(x))\},\ w \in \mathrm{cl}\{B(x, \delta(x))\}\}$.

According to[23] Theorem 14 if (1) is robustly GAS then the origin of its associated difference inclusion is strongly GAS. Moreover, since F is is nonempty, compact, and upper-semicontinuous,[23] Theorem 12 guarantees that the associated difference inclusion is robustly GAS. Therefore, the existence of a differentiable LF for the difference inclusion is guaranteed by[23] Theorem 10.

In the particular case when f is $\boldsymbol{D}_{\mathbf{r}}$-homogeneous of degree $\nu = 1$ we have that F is $\mathbf{r}$-homogeneous of degree $\kappa = 0$ (see e.g.[8],[20],[28], for the definition of homogeneous set-valued maps), this fact is a particular case of[29] Corollary 2.9.